Atomic Units:

$$\hbar = 1 \tag{1}$$

Definitions:

$$\left\langle N\vec{v}^{2}\right\rangle_{\sigma}(E) \equiv \frac{1}{(2\pi)^{3}}\sum_{n}\int_{BZ} d^{3}k\,\delta\left(E - E_{\vec{k},j,n}\right)\vec{v}_{\vec{k},j,n}^{2} \tag{2}$$

$$\left\langle N\right\rangle_{\sigma}(E) \equiv \frac{1}{(2\pi)^{3}}\sum_{n}\int_{BZ} d^{3}k\,\delta\left(E - E_{\vec{k},j,n}\right) \tag{3}$$

$$\gamma_{\sigma}(E) \equiv \frac{\left\langle N\vec{v}^{2}\right\rangle_{\sigma}(E)}{\left\langle N\right\rangle_{\sigma}(E)} \tag{4}$$

$$\vec{v}_{\vec{k},j,\sigma} \equiv \frac{\partial E_{\vec{k},j,\sigma}}{\partial \vec{k}} \tag{5}$$

Dirac Delta Function Properties:

$$\delta\left(af(x)\right) = \frac{1}{|a|}\delta\left(f(x)\right) = \frac{1}{|a|\left|f'(x)\right|}\delta(x) \tag{6}$$

$$\delta\left(x^{2} - a^{2}\right) = \frac{1}{2|a|}\left[\delta(x-a) + \delta(x+a)\right] \tag{7}$$

Electron gas:

$$E_{F,\sigma} = \frac{k_{F,\sigma}^{2}}{2} \qquad v_{k} = k \tag{8}$$

Numerator, denominator and ratio for the electron gas at the Fermi level (one band):

$$\left\langle N\vec{v}^2\right\rangle_\sigma (E_F) \equiv \frac{4\pi}{(2\pi)^3}\int_0^\infty dk\, k^2 \delta\left(\frac{k_F^2}{2}-\frac{k^2}{2}\right)k^2 = \frac{k_F^3}{2\pi^2} \tag{9}$$

$$\left\langle N\right\rangle_\sigma (E_F) \equiv \frac{4\pi}{(2\pi)^3}\int_0^\infty dk\, k^2 \delta\left(\frac{k_F^2}{2}-\frac{k^2}{2}\right) = \frac{k_F}{2\pi^2} \tag{10}$$

$$\gamma_\sigma (E_F) \equiv \frac{\left\langle N\vec{v}^2\right\rangle_\sigma (E_F)}{\left\langle N\right\rangle_\sigma (E_F)} = k_F^2 = v_F^2 = 2E_F \tag{11}$$

The result of Eqtns. (9)-(11) suggests the definition:

$$v_{F,\sigma} = \sqrt{\gamma_\sigma (E_F)} \tag{12}$$

Another equally plausible definition would be:

$$v_{F,\sigma} \equiv \frac{\left\langle N\left|\vec{v}\right|\right\rangle_\sigma (E_F)}{\left\langle N\right\rangle_\sigma (E_F)} \tag{13}$$

with

$$\left\langle N\left|\vec{v}\right|\right\rangle_\sigma (E) \equiv \frac{1}{(2\pi)^3}\sum_n \int_{BZ} d^3k\, \delta\left(E - E_{\vec{k},j,n}\right)\left|\vec{v}_{\vec{k},j,n}\right| \tag{14}$$

Averaging Eqtn. (8.51) of Aschcroft & Mermin over the Fermi surface suggests Eqtn. (12). Explicitly this average is (no electron gas approximation here):

$$\frac{1}{S}\int_S dS \left|\vec{v}\left(\vec{k}_F\right)\right| = \frac{(2\pi)^3}{S}\left\langle N\vec{v}^2\right\rangle (E_F) \tag{15}$$

And the following can be derived (for the electron gas):

$$\frac{(2\pi)^3}{S} = \frac{(2\pi)^3}{4\pi k_F^2} = \frac{2\pi^2}{k_F^2} = \frac{1}{\sqrt{\left(\left\langle N\right\rangle (E_F)\left\langle N\vec{v}^2\right\rangle (E_F)\right)}} \tag{16}$$

Therefore, we see that Eqtn. (14) is the same as the definition adopted in Eqtn. (11). If we can prove Eqtn. (15) in general then I think it is a done deal. Essentially the equality to prove is

$$\left\langle N\left|\vec{v}\right|\right\rangle\left(E_F\right)=\frac{S}{\left(2\pi\right)^3} \qquad ?=? \qquad \sqrt{\left(\left\langle N\right\rangle\left(E_F\right)\left\langle N\vec{v}^{\,2}\right\rangle\left(E_F\right)\right)} \qquad (17).$$

Let me know when you have proved it!   ;)